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Threshold behaviour of discordant voting on the complete graph*

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October 12, 2018

Abstract

Given a connected graph G whose vertices are coloured in some way, a discordant voting process on G is as follows. At each step a pair of adjacent vertices with different colours interact, and one of the vertices changes its colour to match the other one. If eventually all vertices have the same colour, we say a consensus has been reached.

A vertex is discordant if it has a discordant edge, i.e. a neighbour of a different colour. In the general discordant voting process, at each step a discordant vertex u is chosen uniformly at random, and then a discordant edge (u, v) is chosen from among the discordant edges of u , also uniformly at random. With probability β vertex u adopts the colour of vertex v and with probability $1 - \beta$ vertex v adopts the colour of vertex u .

Let T be the number of steps needed to reach consensus. For the complete graph K_n with an initial colouring where half the vertices are red and half blue, then when $\beta = 0$, $\mathbf{E}T = \Theta(n \log n)$, whereas when $\beta = 1$ then $\mathbf{E}T = \Theta(2^n)$. The case $\beta = 1/2$ corresponds to a simple random walk on a path with vertex set $\{0, 1, \dots, n\}$ and has $\mathbf{E}T = n^2/4$. We study the effect of varying β from zero to one, thus revealing the detailed transition from $\mathbf{E}T = \Theta(n \log n)$ to $\mathbf{E}T = \Theta(2^n)$. In terms of β , the transition from $\Theta(n \log n)$ to $\Theta(2^n)$ occurs in a scaling window of width $\mathcal{O}(1/n)$ around $\beta = 1/2$. For any $a > 1$, there is an explicit value of $\beta = 1/2 + \mathcal{O}(\log n/n)$ for which $\mathbf{E}T = \Theta(n^a)$. When $\beta > 1/2$ constant, there is an explicit value $a(\beta)$ such that $\mathbf{E}T = \Theta(a^n)$.

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1 Introduction

We consider a discrete distributed process on graphs called discordant voting. Initially each vertex has one of two colours, red or blue. In a discordant voting process, at each step a pair of adjacent vertices of different colours interact in a predefined manner until all vertices have the same colour. We note that in each round exactly one vertex changes its colour. If at some step all the vertices have the same colour, we say a consensus has been reached. Let T be the number of steps taken to reach consensus, and \mathbf{ET} the expected time to reach consensus.

Discordant voting originated in the complex networks community as a model of social evolution (see e.g. [8]), the colours of the vertices representing the opinions of the distinct social groups. The general version of the model allows *rewiring*. At each step a random discordant edge is chosen. Then either a random endpoint of the chosen edge recolours itself according to the colour of the other endpoint (a voting move) or the edge is deleted and the vertices reconnect elsewhere (a rewiring move). This serves as a simple model of social interaction in which vertices either change their opinions or their friends. Rewiring has been studied under various assumptions by several authors including Durrett et al. [4], [5], and Basu and Sly [2].

Given the current vertex colours, an edge is discordant if its endpoints are different colours, and a vertex is discordant if it has at least one discordant edge. There are several different ways to carry out a discordant voting step. Pick a random discordant vertex and push its colour to a random discordant neighbour. Pick a random discordant vertex and pull the colour of a random discordant neighbour. Pick a random endpoint of a random discordant edge and push the colour to the other end point. We refer to these choices as push, pull and oblivious voting respectively.

The rewiring papers above all use the oblivious voting model. In the absence of rewiring, the performance of the oblivious voting process is independent of graph structure and is thus the same for any connected n -vertex graph. It depends only on the initial number of vertices of each colour (R_0, B_0). The reason for this is as follows. At any step, and whatever discordant edge is chosen, the number of blue vertices in the graph increases (resp. decreases) by one with probability $1/2$. This is equivalent to an unbiased random walk on the line $(0, 1, \dots, n)$ with absorbing barriers, and starting from $R_0 = r$ red vertices. Thus $\mathbf{ET}(\text{Oblivious}) = r(n - r)$ (see Feller [6, XIV.3]). In the worst case, starting with an equal number of red and blue vertices ($R_0, B_0 = n/2$) the oblivious protocol has $\mathbf{ET} = n^2/4$ for any connected graph. In this paper we always assume worst case initial colouring ($R_0 = B_0 = n/2$) for comparison purposes.

In stark contrast to the oblivious protocol, the push and pull protocols can exhibit very different expected times to consensus, which depend strongly on the structure of underlying graph in question (see [3] for more detail). For example, on the complete graph K_n , starting from an initial colouring where half the vertices are red and half blue, then $\mathbf{ET}(\text{Push}) = \Theta(n \log n)$, whereas $\mathbf{ET}(\text{Pull}) = \Theta(2^n)$.

In order to study this transition from the $\Theta(n \log n)$ expected consensus time for the push protocol to the $\Theta(2^n)$ expected consensus time for the pull protocol, we introduce a parameter $\beta \in [0, 1]$ which measures the probability of a pull move at any step. When $\beta = 1/2$ then push and pull are equally probable. For regular graphs $\beta = 1/2$ is equivalent to the oblivious protocol giving a $\Theta(n^2)$ consensus time. The push, oblivious and pull models are thus particular cases of the β -Push-Pull model, with $\beta = 0, 1/2, 1$ respectively.

The question is, as β varies from zero to one, how exactly does the β -Push-Pull process make the transition in \mathbf{ET} from order $n \log n$ (push: $\beta = 0$), to order n^2 (oblivious: $\beta = 1/2$), and finally to order 2^n (pull: $\beta = 1$). Theorems 1, 2 give the answer, which we briefly summarize. The push protocol favors the majority, whereas the pull acts in favor of the minority opinion. Tiny changes in the balance between these actions have a profound effect on the time to consensus.

In computer science there is a focus on the distinction between processes with a polynomial running time $T = \mathcal{O}(n^c)$ and those with an exponential running time $T = \Omega(c^n)$. Crudely, if we only consider discordant voting for constant values of β , then for $\beta < 1/2$ the (expected) run time is $\Theta(n \log n)$, when $\beta = 1/2$ the run time is $\Theta(n^2)$, and for $\beta > 1/2$, the run time is exponential. Using Theorems 1 and 2 we can find a β corresponding to any place in the complexity hierarchy from order $n \log n$ to 2^n .

In more detail, if $\beta \leq 1/2 + \mathcal{O}(\log n/n)$ then $\mathbf{ET}(\beta) = \Theta(n^c)$ for some $c > 1$. The value of $\beta(c)$ needed to obtain this value of c is given by Corollary 3. The transition at $\beta = 1/2$ is not symmetric. For example choosing $\varepsilon = \log n/n$ then $\mathbf{ET}(1/2 - \varepsilon) = \Theta(n^2 \log \log n / \log n)$ and $\mathbf{ET}(1/2 + \varepsilon) = \Theta(n^3 / (\log n)^{3/2})$.

Next consider $\beta = 1/2 + \varepsilon$ where $\varepsilon = \omega(\log n/n)$ but $\varepsilon < 1/2$. The expected run time makes a transition from polynomial to exponential, given by $\log(\mathbf{ET}(\beta))/n = \varepsilon + \mathcal{O}(\varepsilon^3) + \mathcal{O}(\log n/n)$, see Theorem 2(ii)-(iii). If $\beta < 1$ constant, then $\mathbf{ET}(\beta)/2^n = o(1)$ with a transition to $\mathbf{ET} = \Theta(2^n)$ when $\beta = 1 - o(1/n)$.

Theorem 1 *Let $0 \leq \beta \leq 1/2$. Let \mathbf{ET} be the expected time to reach consensus in the β -Push-Pull process on the complete graph K_n , starting from $R_0, B_0 = n/2$.*

- (i) *If $\beta \in [0, 1/2)$ independent of n , then $\mathbf{ET} = \Theta(n \log(n))$.*
- (ii) *If $\beta = 1/2 - \varepsilon$ where $\varepsilon > \frac{1}{n}$, then $\mathbf{ET} = \Theta\left(\frac{n}{\varepsilon} \log(n\varepsilon)\right)$.*
- (iii) *If $\beta = 1/2 - \varepsilon$ where $\varepsilon = \mathcal{O}(\frac{1}{n})$, then $\mathbf{ET} = \Theta(n^2)$.*

Theorem 2 *Let $1/2 \leq \beta \leq 1$. Let \mathbf{ET} be the expected time to reach consensus in the β -Push-Pull process on the complete graph K_n , starting from $R_0, B_0 = n/2$.*

- (i) *If $\beta = 1/2 + \varepsilon$ where $\varepsilon = \mathcal{O}(\frac{1}{n})$, then $\mathbf{ET} = \Theta(n^2)$.*

(ii) If $\beta = 1/2 + \varepsilon$ where $\varepsilon > 1/n$ but $\varepsilon = o(1)$, then

$$\begin{aligned} \mathbf{ET} &= \Theta(\sqrt{n}/\varepsilon^{3/2}) \exp\left(\frac{n}{2\varepsilon} \sum_{j \geq 1} \frac{1}{(2j-1)(2j)} (2\varepsilon)^{2j}\right) \\ &= \Theta(\sqrt{n}/\varepsilon^{3/2}) \exp(n\varepsilon(1 + \mathcal{O}(\varepsilon^2))). \end{aligned} \quad (1)$$

(iii) If $\beta = 1/2 + \varepsilon = 1 - \delta$ where ε is a constant independent of n , then

$$\begin{aligned} \mathbf{ET} &= \Theta(\sqrt{n}) \left((1 + 2\varepsilon)^{\frac{1}{2} + \varepsilon} (1 - 2\varepsilon)^{\frac{1}{2} - \varepsilon} \right)^{\frac{n}{2\varepsilon}} \\ &= \Theta(\sqrt{n}) (2(1 - \delta)^{1 - \delta} \delta^\delta)^{\frac{n}{1 - 2\delta}}. \end{aligned} \quad (2)$$

(iv) If $\beta = 1 - \delta$ where $\delta = o(1)$ but $\delta \geq 1/n\omega$ where $\omega \rightarrow \infty$ then

$$\mathbf{ET} = \Theta(\sqrt{n\delta}) 2^n (4\delta/e)^{n\delta}. \quad (3)$$

(v) If $\beta = 1 - \delta$ where $\delta \leq 1/n\omega$ where $\omega \rightarrow \infty$ then

$$\mathbf{ET} = \Theta(1) 2^n \exp(-\delta n \log n). \quad (4)$$

(vi) If $\beta = 1 - \delta$ where $\delta = \mathcal{O}(1/n \log n)$ then $\mathbf{ET} = \Theta(2^n)$.

We see that the transition, in terms of β , from $\Theta(n \log n)$ to $\Theta(n^2)$ and beyond occurs in a scaling window of width $o(1)$ around $\beta = 1/2$. Using Theorems 1 and 2 we can find a β corresponding to any place in the complexity hierarchy from order $n \log n$ to 2^n . As an example of this, for any $a > 1$, the value of $\beta \sim 1/2 + o(1)$ giving $\mathbf{ET} = \Theta(n^a)$, is given by the following corollary.

Corollary 3 *To obtain an expected completion time of $\Theta(n^a)$, $a > 1$ constant, choose β as follows. Let $c > 0$ constant, then*

- (i) *If $1 < a < 2$ then put $\beta = 1/2 - \varepsilon$ where $\varepsilon = c(\log n)/n^{a-1}$.*
- (ii) *If $a = 2$ put $\beta = 1/2 + \varepsilon$ where $|\varepsilon| = c/n$.*
- (iii) *if $a > 2$ then put $\beta = 1/2 + \varepsilon$ where $\varepsilon = ((a - 2) \log n + (3/2) \log \log n)/n$.*

2 Birth-and-Death chains

A Markov chain $(X_t)_{t \geq 0}$ is said to be a Birth-and-Death chain on state space $S = \{0, \dots, N\}$ if given $X_t = i$ then the possible values of X_{t+1} are $i+1$, i or $i-1$ with probability p_i , r_i and q_i respectively. Note that $q_0 = p_N = 0$. In this section we assume that $r_i = 0$,

$p_0 = 1$, $q_N = 1$, $p_i > 0$ for $i \in \{0, \dots, N-1\}$ and $q_i > 0$ for $i \in \{1, \dots, N\}$. We denote $\mathbf{E}_i Y$ the expected value of random variable Y when the chain starts in i (i.e., $X_0 = i$). Finally, we define the (random) hitting time of state i as $T_i = \min\{t \geq 0 : X_t = i\}$.

We summarize the results we require on Birth-and-Death chains (see Peres, Levin and Wilmer [7, chapter 2.5]).

Say that a probability distribution π satisfies the detailed balance equations, if

$$\pi(i)P(i, j) = \pi(j)P(j, i), \text{ for all } i, j \in S. \quad (5)$$

Birth-and-Death chains with $p_i = P(i, i+1)$, $q_i = P(i, i-1)$ can be shown to satisfy the detailed balance equations. Thus we can solve in terms of $\pi(0)$,

$$\pi(1) = \frac{\pi(0)p_0}{q_1} \text{ and } \pi(i) = \pi(0) \frac{p_0 \cdots p_{i-1}}{q_1 \cdots q_i} \text{ for } i \geq 2. \quad (6)$$

If we normalize π , we obtain a stationary distribution. The normalization constant is

$$\pi(0) = \left(1 + \frac{p_0}{q_1} + \sum_{j=2}^n \frac{p_0 \cdots p_{j-1}}{q_1 \cdots q_j} \right)^{-1}. \quad (7)$$

As the Markov chain is recurrent, the stationary distribution satisfies

$$\pi(i) = \frac{1}{\mathbf{E}_i(T_i^+)},$$

where $\mathbf{E}_i(T_i^+)$ is the expected first return time to state i (on exit from state i). It follows from this, (see e.g. [7]) that

$$\mathbf{E}_{i-1}T_i = \frac{1}{q_i\pi(i)} \sum_{k=0}^{i-1} \pi(k) \quad (8)$$

An equivalent formulation (see [7]) is $\mathbf{E}_0T_1 = 1/p_0 = 1$ and in general

$$\mathbf{E}_{i-1}T_i = \sum_{k=0}^{i-1} \frac{1}{p_k} \frac{q_{k+1} \cdots q_{i-1}}{p_{k+1} \cdots p_{i-1}}, \quad \text{for } i \in \{1, \dots, N\}. \quad (9)$$

In writing this expression we follow the convention that if $k = i-1$ then $\frac{q_{k+1} \cdots q_{i-1}}{p_{k+1} \cdots p_{i-1}} = 1$ so that the last term is $1/p_{i-1}$. Note also that the final index k on p_k is $k = N-1$, i.e. we never divide by $p_N = 0$.

Starting from state 0, let T_M be the number of transitions needed to reach state M for the first time. For any $M \leq N$, we have that $\mathbf{E}_0T_M = \sum_{i=1}^M \mathbf{E}_{i-1}T_i$. For example, $\mathbf{E}_0T_1 = \frac{1}{p_0} = 1$ and $\mathbf{E}_0T_2 = 1 + \frac{1}{p_1} + \frac{q_1}{p_0p_1}$ etc. Thus, for $M \geq 1$

$$\mathbf{E}_0T_M = \sum_{i=1}^M \mathbf{E}_{i-1}T_i = \sum_{i=1}^M \sum_{k=0}^{i-1} \frac{1}{p_k} \prod_{j=k+1}^{i-1} \frac{q_j}{p_j}. \quad (10)$$

To find the time to consensus we can consider $Y_t = \max\{R_t, B_t\}$. As $Y_t \geq n/2$, we can write $Y_t = n/2 + i$. We define two Birth-and-Death chains which underlie our analysis. The chains have states $\{0, 1, \dots, i, \dots, N\}$ where $N = n/2$ (assume $n \geq 2$ even). The transition probabilities from state i given by $P(i, i+1)$, $Q(i, i+1) = 1 - P(i, i+1)$.

Push Chain. Let Y_t be the state occupied by the push chain at step $t \geq 0$. The transition probability $P_i = P(i, i+1)$ from $Y_t = i$, is given by

$$P_i = \begin{cases} 1, & \text{if } i = 0 \\ 1/2 + i/n, & \text{if } i \in \{1, \dots, n/2 - 1\} \\ 0, & \text{if } i = n/2 \end{cases} \quad (11)$$

Pull Chain. Let \bar{Y}_t be the state occupied by the pull chain at step $t \geq 0$. Given that $\bar{Y}_t = i$, the transition probability $\bar{P}_i = \bar{P}(i, i+1)$ is given by

$$\bar{P}_i = \begin{cases} 1, & \text{if } i = 0 \\ 1/2 - i/n, & \text{if } i \in \{1, \dots, n/2 - 1\} \\ 0, & \text{if } i = n/2 \end{cases} \quad (12)$$

For $1 \leq i \leq N - 1$ the pull chain is the push chain with the probabilities reversed, i.e. $\bar{P}_i = Q_i$.

The β -Push-Pull process is a mixture of the push and pull chains with transition probability $p_i = (1 - \beta)P_i + \beta\bar{P}_i$. The transition probabilities of $Z_t = \max\{R_t, B_t\}$ for the β -Push-Pull process are as follows. Let $p_i = \mathbf{P}(Z_{t+1} = Z_t + 1 \mid Z_t = i)$, then

$$p_i = \begin{cases} 1, & \text{if } i = 0 \\ 1/2 + (1 - 2\beta)i/n, & \text{if } i \in \{1, \dots, n/2\} \\ 0, & \text{if } i = n/2 \end{cases} \quad (13)$$

If $n = 2N + 1$ is odd then the only difference is that the states i of the chain are $i \in \{1, \dots, N + 1\}$, and states $1, N + 1$ are reflecting, rather than $0, N$. For convenience of notation (i.e to avoid writing $N = \lceil n/2 \rceil$) we analyse the case $n = 2N$.

Let $M \in \{0, 1, \dots, n/2\}$ be the start position of Z_t . Let $T_N = T_N(\beta, M)$ be the time taken to reach position N starting from M , and let $\mathbf{E}_M T_N$ be the expectation of T_N . The following natural result, proved by coupling is used extensively in the proofs.

Lemma 4 *Let $0 \leq \beta \leq \beta' \leq 1$, and $N \geq M$. Then $\mathbf{E}_M T_N(\beta) \leq \mathbf{E}_M T_N(\beta')$.*

Proof. Let Π, Π' be the respective particles making the walks from start position $M = \max(R_0, B_0)$ and let i, i' be the current positions of the particles.

At each step sample r uniformly from $[0, 1]$. If $r \leq p_i(\beta)$, Π moves right and otherwise moves left. If $r \leq p_{i'}(\beta')$, Π' moves right and otherwise moves left.

At the start $i = i' = M$. The particles stay identically coupled until Π moves right and Π' moves left. Thus at any step $t \geq 0$, $(i_t - i'_t) = 2k_t \geq 0$, and so $T_N(\beta) \leq T_N(\beta')$. \square

3 Analysis of the case $\beta \leq 1/2$

3.1 A general $n \log(n)$ estimate

We develop a general recipe to obtain $\Theta(n \log(n))$ estimates of the time to reach consensus. First, we show the upper bound.

Theorem 5 *Consider a birth and death process over $\{0, \dots, n\}$ with $p_0 = q_n = 1$. If the following conditions hold:*

1. *For all $k \in \{1, \dots, n-2\}$, $\frac{p_k}{q_k} \leq \frac{p_{k+1}}{q_{k+1}}$.*
2. *There exist a constant $C_1 > 0$ such that for all $k \in \{1, \dots, n-1\}$:*

$$\frac{1}{p_k - q_k} \leq C_1 \frac{n}{k}.$$

3. *For all $k \in \{1, \dots, n-2\}$ exists a constants $C_2 > 0$ such that*

$$\frac{p_{k+1}}{p_k} \leq C_2.$$

Then

$$\mathbf{E}_0(T_n) \leq C_1 C_2 n \log(n) + \mathcal{O}(n).$$

Proof. To begin with, by changing order of summation in (10) it follows that

$$E_0 T_n = \sum_{i=1}^n \frac{1}{p_k} \sum_{k=0}^{i-1} \frac{1}{p_k} \prod_{j=k+1}^{i-1} \frac{q_j}{p_j} = \sum_{k=0}^{n-1} \frac{1}{p_k} \sum_{i=k+1}^n \frac{q_{k+1} \cdots q_{i-1}}{p_{k+1} \cdots p_{i-1}}. \quad (14)$$

By the first condition the ratios q_j/p_j are decreasing, and second condition implies $\frac{q_{k+1}}{p_{k+1}} < 1$. Thus

$$E_0 T_n \leq \sum_{k=0}^{n-1} \frac{1}{p_k} \sum_{m=0}^{\infty} \left(\frac{q_{k+1}}{p_{k+1}} \right)^m.$$

Adding things up,

$$\sum_{k=0}^{n-1} \frac{1}{p_k} \sum_{m=0}^{\infty} \left(\frac{q_{k+1}}{p_{k+1}} \right)^m = \sum_{k=0}^{n-1} \frac{1}{p_k} \frac{1}{1 - \frac{q_{k+1}}{p_{k+1}}} = \sum_{k=0}^{n-1} \frac{p_{k+1}}{p_k} \frac{1}{p_{k+1} - q_{k+1}}.$$

Finally, by using the upper bounds given in the second and third conditions we obtain the claimed result. \square

Theorem 6 Consider a birth and death process over $\{0, \dots, n\}$. If the following conditions hold:

1. For all $k \geq 1$, $p_k \leq p_{k+1}$ and $q_k \geq q_{k+1}$.
2. There exist a constant C_1 such that for all $k \in \{1, \dots, n\}$,

$$\frac{1}{p_k - q_k} \geq C_1 \frac{n}{k}.$$

3. There exist a constant C_2 such that for all $k \in \{1, \dots, n\}$,

$$\frac{q_k}{p_k} \leq \left(1 - \frac{C_2 k}{n} \right),$$

then, asymptotically,

$$\mathbf{E}_0(T_n) \geq \left(\frac{C_1}{2} \right) n \log \left(\frac{n}{2} \right) + \mathcal{O}(n).$$

Proof. Using (14) we have

$$E_0 T_n = \sum_{i=1}^n \frac{1}{p_i} \sum_{k=0}^{i-1} \prod_{j=k+1}^{i-1} \frac{q_j}{p_j} \geq \sum_{i=1}^n \frac{1}{p_i} \sum_{k=0}^{i-1} \left(\frac{q_i}{p_i} \right)^{i-k-1} = \sum_{i=1}^n \frac{1}{p_i - q_i} \left(1 - \left(\frac{q_i}{p_i} \right)^i \right).$$

Thus

$$\mathbf{E}_0 T_n \geq \sum_{i=\lfloor \sqrt{n} \rfloor}^n C_1 \frac{n}{i} \left(1 - \left(1 - \frac{C_2 i}{n} \right)^i \right) \geq \sum_{i=\lfloor \sqrt{n} \rfloor}^n C_1 \frac{n}{i} - \sum_{i=\lfloor \sqrt{n} \rfloor}^n C_1 \frac{n}{i} \exp \left\{ -\frac{C_2 i^2}{n} \right\}. \quad (15)$$

The first term of the sum is $C_1 n (H(n) - H(\lfloor \sqrt{n} \rfloor))$ with $H(n)$ the n -th harmonic number. The second term is more interesting, a simple bound gives us

$$\begin{aligned} \sum_{i=\lfloor \sqrt{n} \rfloor}^n C_1 \frac{n}{i} \exp \left\{ -\frac{C_2 i^2}{n} \right\} &\leq C_1 \sqrt{n} \sum_{i=\lfloor \sqrt{n} \rfloor}^n \exp \left\{ -\frac{C_2 i^2}{n} \right\} \\ &\leq C_1 \sqrt{n} \int_0^n \exp \left\{ -\frac{C_2 x^2}{n} \right\} dx \leq n \frac{C_1 \sqrt{\pi}}{2\sqrt{C_2}}. \end{aligned}$$

Our final estimate becomes

$$\mathbf{E}_0 T_n \geq C_1 n (H(n) - H(\lfloor \sqrt{n} \rfloor)) - n \frac{C_1 \sqrt{\pi}}{2\sqrt{C_2}} \sim \frac{C_1}{2} n \log(n) + \mathcal{O}(n).$$

\square

4 Case $\beta \in (0, 1/2)$: Proof of Theorem 1

4.1 Proof of Theorem 1.i

The proof of the item i of Theorem 1 is just an application of the result obtained in section 3.1. We apply the above theorems to the Markov chain $(Z_t)_{t \geq 0}$ whose transition probabilities are given by equation (13).

First of all, we have to take with the endpoints of the chains. The Markov chain $(Z_t)_{t \geq 0}$ moves from 0 to $n/2$ while the chains of Theorems 5 and 6 move from 0 to n .

We start with β -Push-Pull model for fixed $\beta \in [0, \frac{1}{2})$. Note that when $\beta = 0$, we recover Push model. In that case it is straightforward to verify the conditions of theorem 5. Note that p_k/q_k is an increasing function on k for $k \in \{1, \dots, n/2 - 2\}$ which is the first condition. Also note that

$$\frac{1}{p_k - q_k} = 2(1 - 2\beta) \frac{n}{k},$$

that make us take $C_1 = 2(1 - 2\beta)$ for the second condition. Finally $p_{k+1}/p_k = 1 + \mathcal{O}(1/n)$ for $k \in \{1, \dots, n - 2\}$. Those three conditions give us that β -Push-Pull model satisfies

$$\begin{aligned} \mathbf{E}_0 T_{n/2} &\leq \frac{1}{2(1 - 2\beta)} (1 + \mathcal{O}(1/n)) \frac{n}{2} \log(n/2) + \mathcal{O}(n) \\ &= \frac{1}{4(1 - 2\beta)} (1 + \mathcal{O}(1/n)) n \log(n/2) + \mathcal{O}(n). \end{aligned}$$

For a lower bound we use theorem 6. The first condition is true, for the second condition we use $C_1 = 2(1 - 2\beta)$. The last condition can be checked with $C_2 = 4(1 - 2\beta)$, obtaining that

$$\mathbf{E}_0 T_{n/2} \geq \frac{1}{8(1 - 2\beta)} n \log(n/2) + \mathcal{O}(n).$$

Note the two above inequalities give us very good estimates, indeed the lower and the upper bound are equal up to the multiplicative constant of 2.

4.2 Proof of Theorem 1.ii and 1.iii

We start by saying that the case $\beta = 1/2$ makes the Birth-and-Death chain $(Z_t)_{t \geq 0}$ (with transition probabilities given by equation (13)) a simple random walk in a line, thus the time to reach consensus is $\Theta(n^2)$ (See [7] for details)

Consider $\varepsilon = \varepsilon_n \rightarrow 0$, $\varepsilon > 0$ and choose $\beta = 1/2 - \varepsilon$. We can assume that $\beta > 0$ for every n . Define $\delta = \frac{1}{\varepsilon}$ and assume that $\delta < n/2$. To simplify notation, we define $N = n/2$.

Theorem 7 Let $\varepsilon = \varepsilon_n$, $\varepsilon > 0$ and $\varepsilon \rightarrow 0$, and $\delta = \delta_n$ be two constants such that $\varepsilon = \frac{1}{\delta}$ and $\delta < N/2$. Then, for large n , β -Push-Pull model with $\beta = \frac{1}{2} - \varepsilon$ we have

$$\mathbf{E}_0 T_{n/2} \leq (2 + o(1)) \exp(8\varepsilon) \frac{n}{\varepsilon} (\log(n\varepsilon)) + \mathcal{O}(n\delta),$$

and there exists a constant $K = K_\varepsilon = \Theta(1)$ such that

$$\mathbf{E}_0 T_{n/2} \geq K \frac{n}{\varepsilon} \log\left(\frac{n}{2}\varepsilon\right).$$

The proof of Theorem 1.ii and 1.iii is a direct consequence of Theorem 7, indeed, note that for $\varepsilon = o(1/n)$ we can apply directly theorem 7. For $\mathcal{O}(1/n) = \varepsilon > 2/N = 4/n$ we obtain a $\Theta(n^2)$ estimate. For $4/n \geq \varepsilon$ we can couple the process between a process with $\varepsilon = 5/n$ and $\varepsilon = 0$. In both cases the process take $\Theta(n^2)$ steps to reach state N .

We proceed to prove Theorem 7.

Proof.

We are going to find estimates for equation (14). Note that $p_i = 1/2 + 2\varepsilon i/n$ and $q_i = 1 - p_i$, hence

$$\frac{q_i}{p_i} = \exp\left(\log\left(1 - \frac{4\varepsilon i}{n}\right) - \log\left(1 + \frac{4\varepsilon i}{n}\right)\right), \quad (16)$$

but $\log(1+x) = \sum_{k \geq 1} (-1)^{k+1} \frac{x^k}{k}$ for $|x| < 1$, but since i is at most $\frac{n}{2}$ and $\varepsilon < 1/2$ we can apply the logarithm expansion on equation (16), hence

$$\frac{q_i}{p_i} = \exp\left(-2 \sum_{k \geq 1} \frac{(\theta i)^{2k-1}}{2k-1}\right), \quad (17)$$

where $\theta = \frac{4\varepsilon}{n}$. Note that $\theta i \leq 2\varepsilon$.

We estimate the sum inside equation (17),

$$\begin{aligned} -2 \sum_{k \geq 1} \frac{(\theta i)^{2k-1}}{2k-1} &= -2\theta i \left(1 + \frac{(\theta i)^2}{3} + \frac{(\theta i)^4}{5} + \frac{(\theta i)^6}{7} + \dots\right) \\ &\geq -2\theta i \left(1 + \frac{(2\varepsilon)^2}{3} + \frac{(2\varepsilon)^4}{5} + \frac{(2\varepsilon)^6}{7} + \dots\right) \\ &\geq -\frac{2\theta i}{1 - 4\varepsilon^2}. \end{aligned}$$

Noticing that the above sum is clearly upper bounded by $-2\theta i$, we conclude that for every $\varepsilon < 1/2$

$$\frac{q_i}{p_i} \in \left[\exp\left(-\frac{2\theta i}{1 - 4\varepsilon^2}\right), \exp(-2\theta i)\right], \quad (18)$$

even for a ε constant.

Upper Bound. We begin by finding a good upper bound for equation (14). Let start by replacing equation (18) into equation (14) and the fact that $\frac{1}{p_i} = 2 + o(1)$, then we obtain

$$\begin{aligned}
 E_0 T_N &\leq (2 + o(1)) \sum_{k=0}^{N-1} \sum_{i=k+1}^N \exp \left(-2\theta \sum_{j=k+1}^{i-1} j \right) \\
 &= (2 + o(1)) \sum_{k=0}^{N-1} \sum_{i=k+1}^N \exp \left(-\theta(i^2 - k^2 - k - i) \right) \\
 &= (2 + o(1)) \sum_{k=0}^{N-1} \sum_{i=k+1}^N \exp \left(-\theta(i^2 - k^2) \right) \exp(\theta(i + k)) \\
 &\leq (2 + o(1)) \exp(8\varepsilon) \sum_{k=0}^{N-1} \sum_{i=k+1}^N \exp \left(-\theta(i^2 - k^2) \right). \tag{19}
 \end{aligned}$$

To deal with the sum of equation (19) we make the change of variable $l = i - k$, obtaining

$$\begin{aligned}
 \sum_{k=0}^{N-1} \sum_{i=k+1}^N \exp \left(-\theta(i^2 - k^2) \right) &= \sum_{k=0}^{N-1} \sum_{l=1}^{N-k} \exp \left(-\theta(l^2 + 2lk) \right) \\
 &= \sum_{l=1}^N \exp(-\theta l^2) \sum_{k=0}^{N-l} \exp(-2\theta lk) \tag{20}
 \end{aligned}$$

Using an integral approximation, we obtain

$$\begin{aligned}
 \sum_{l=1}^N \exp(-\theta l^2) \sum_{k=0}^{N-l} \exp(-2\theta lk) &= \sum_{l=1}^N \exp(-\theta l^2) \int_0^{N-l} \exp(-2\theta lk) dk + \mathcal{O}(N) \\
 &= \mathcal{O}(n) + \frac{1}{2\theta} \sum_{l=1}^N \exp(-\theta l^2) \frac{1}{l} (1 - \exp(-2\theta l(N-l))) \\
 &= \mathcal{O}(n) + \frac{1}{2\theta} \sum_{l=1}^N \exp(-\theta l^2) \frac{1}{l} (1 - \exp(-4\varepsilon l)).
 \end{aligned}$$

We separate the sum of equation (20) in three parts; $1 \leq l \leq \lfloor \delta \rfloor$, $\delta < l \leq \sqrt{N\delta}$ and $\lceil \sqrt{N\delta} \rceil < l \leq N$. The function $f(x) = (1 - e^{-4\varepsilon x})/x$ is monotone decreasing for $x \geq 0$. By a series expansion, $f(0) = 4\varepsilon$. Thus

$$\sum_{l=1}^{\lfloor \delta \rfloor} f(l) \leq 4\varepsilon \delta = 4, \quad \sum_{l=\delta+1}^{\sqrt{N\delta}} f(l) \leq \sum_{l=\delta+1}^{\sqrt{N\delta}} \frac{1}{l} \leq \log \sqrt{N/\delta} + \mathcal{O}(1).$$

Therefore, as $N = n/2$, $\delta = 1/\varepsilon$ and $\theta = 4\varepsilon/n$,

$$\sum_{l=1}^{\sqrt{N\delta}} \exp(-\theta l^2) \sum_{k=0}^{N-l} \exp(-2\theta lk) \leq \mathcal{O}(n) + \mathcal{O} \left(\frac{4 + \log \sqrt{n\varepsilon}}{\theta} \right) = \mathcal{O} \left(\frac{n}{\varepsilon} \log(n\varepsilon) \right). \tag{21}$$

Last part is to sum from $\sqrt{N\delta}$ to N . As $\varepsilon \geq 4/n$, then $l \geq \lceil \sqrt{N\delta} \rceil \geq 2$, and $\theta l^2 \geq 2$

$$\begin{aligned} \sum_{l=\lceil \sqrt{N\delta} \rceil}^N \exp(-\theta l^2) \sum_{k=0}^{N-l} \exp(-2\theta lk) &\leq \mathcal{O}(n) + \frac{1}{2\theta} \sum_{l=\lceil \sqrt{N\delta} \rceil}^N \frac{\exp(-\theta l^2)}{l} \\ &\leq \mathcal{O}(n) + \mathcal{O}\left(\frac{1}{\theta}\right) \int_2^\infty \frac{e^{-k^2/2}}{k} dk = \mathcal{O}\left(\frac{1}{\theta}\right). \end{aligned} \quad (22)$$

Lower Bound We proceed to compute a lower bound for equation (14). Let $\theta' = \theta/(1 - 4\varepsilon^2)$. As in the upper bound, replace equation (18) into equation (14) and use that $\frac{1}{p_i} \geq 2$ and perform the change of variable $l = i - k$ to obtain

$$\begin{aligned} E_0 T_N &\geq 2 \sum_{k=0}^{N-1} \sum_{i=k+1}^N \exp\left(-2\theta' \sum_{j=k+1}^{i-1} j\right) \\ &\geq 2 \sum_{k=0}^{N-1} \sum_{i=k+1}^N \exp(-\theta'(i^2 - k^2)) \\ &= 2 \sum_{l=1}^N \exp(-\theta' l^2) \sum_{k=0}^{N-l} \exp(-2\theta' lk). \end{aligned} \quad (23)$$

To obtain a lower bound we just consider the sum of equation (23) when l goes from $\delta/(1 - 4\varepsilon^2)$ to $\sqrt{n\delta}$. Noting that $\theta' l^2 \leq 4/(1 - 4\varepsilon^2) \leq 5$ assuming $\varepsilon \leq 1/5$. Thus, a lower bound is given by

$$\begin{aligned} \sum_{l=1}^N \exp(-\theta' l^2) \sum_{k=0}^{N-l} \exp(-2\theta' lk) &\geq \sum_{l=\delta}^{\sqrt{n\delta}} \exp(-\theta' l^2) \sum_{k=0}^{N-l} \exp(-2\theta' lk) \\ &\geq \sum_{l=\delta}^{\sqrt{n\delta}} e^{-5} \int_{k=0}^{N-l} \exp(-2\theta' lk) \\ &\geq \frac{e^{-5}}{2\theta'} \sum_{l=\delta}^{\sqrt{n\delta}} \frac{1}{l} (1 - \exp(-2\theta' l(N-l))) \\ &\geq \frac{e^{-5}}{2\theta'} (1 - e^{-2}) \sum_{l=\delta}^{\sqrt{n\delta}} \frac{1}{l}. \end{aligned}$$

The last line follows because

$$2\theta' l(N-l) \geq 2 \cdot \frac{4\varepsilon}{n} \delta(N-\delta) \geq 2.$$

Thus for some $C > 0$ constant

$$\mathbf{E}_0 T_N \geq \frac{C}{\theta} \log \sqrt{N/\delta} = \Theta\left(\frac{n}{\varepsilon} \log \frac{n}{\varepsilon}\right).$$

□

5 Analysis of the case when $\beta \in (1/2, 1)$

We start by providing the necessary tools. In all this section we assume that n is divisible by 4. Let $N = n/2$ and define the commute time $C[0, N] = \mathbf{E}_0 T_N + \mathbf{E}_N T_0$. Since $\beta \in (1/2, 1)$ we have that the bias of the random walk is towards 0, then $\mathbf{E}_N T_0 \leq \mathbf{E}_0 T_N$. Therefore, we get

$$\mathbf{E}_0 T_N \leq C[0, N] \leq 2\mathbf{E}_0 T_N, \quad (24)$$

hence, $C[0, N]$ is a good estimate for $\mathbf{E}_0 T_N$. To compute $C[0, N]$ we use the following lemma, whose proof is standard and can be found in, for example, [1].

Lemma 8 *Let a, b be two states of a ergodic Markov chain, then*

$$C[a, b] = \frac{1}{\pi_b \mathbf{P}_b(T_a < T_b^+)}.$$

Applying Lemma 8 to $C[0, N]$ we obtain a way to compute it. We need to estimate $\pi(N)$ and $\mathbf{P}_N(T_0 < T_N^+)$.

6 Background material

6.1 Estimation of $\mathbf{P}_N(T_0 < T_N^+)$.

Define $(X_t)_{t \geq 0} \sim \text{BD}(p, N)$ as the random walk (birth-and-death process) on $\{0, \dots, N\}$ with reflecting barriers at 0 and N and for $i \in \{1, \dots, N-1\}$ we have $\mathbf{P}(X_{t+1} = i+1 | X_t = i) = p$ and $\mathbf{P}(X_{t+1} = i-1 | X_t = i) = q = 1-p$. We define $g_N(p)$ the probability that starting at N the Markov chain $(X_t)_{t \geq 0} \sim \text{BD}(p, N)$ reach state 0 before returning to N . For the following result see [6] Ch. XIV, equation (2.4).

Lemma 9 *For $p \neq q$ we have $g_N(p) = \frac{1-(p/q)}{1-(p/q)^N} \geq q-p$.*

Lemma 10 *For $\beta \in (1/2, 1)$ we have*

$$g_N(1-\beta) \geq \mathbf{P}_N(T_0 < T_N^+) \geq \frac{1}{2} g_{N/2}(3/4 - \beta/2).$$

Proof. For the upper bound consider the process $(X_t)_{t \geq 0} \sim \text{BD}(1-\beta, N)$. Let $\beta = 1/2 + \varepsilon$. Then $p_i = 1/2 + (1-2\beta)/n = 1/2 - \varepsilon(2i/n)$. As $1-\beta = 1/2 - \varepsilon$, we have $p_i > 1-\beta$ for all $i \in \{1, \dots, N-1\}$. Thus we can couple (X_t) with (Z_t) such that $\mathbf{P}(X_t \leq Z_t, \forall t \geq 0 | X_0 = Z_0 = N) = 1$. In particular we get that $g_N(1-\beta) \geq \mathbf{P}_N(T_0 < T_N^+)$.

For the lower bound choose $M = N/2$. By the Markov property we have

$$\mathbf{P}_N(T_0 < T_N^+) = \mathbf{P}_N(T_M < T_N^+) \mathbf{P}_M(T_0 < T_N).$$

To estimate $\mathbf{P}_N(T_M < T_N^+)$ we restrict our chain to the states $\{M, \dots, N\}$, with a new reflecting barrier at M . In the restricted chain we have $p_i \in [1/2 - \varepsilon, 1/2 - \varepsilon/2]$, for all states $i \in \{M + 1, \dots, N - 1\}$. Thus $p_i \leq 1/2 - \varepsilon/2 = 3/4 - \beta/2$.

The lower bound then follows from a coupling argument between (Z_t) on $\{M, \dots, N\}$ and the chain $\text{BD}(3/4 - \beta/2, N - M)$ and the observation that, since the bias of the chain (Z_t) is towards 0, we have $\mathbf{P}_M(T_0 < T_N) \geq 1/2$. \square

Putting Lemma 9 into Lemma 10 we obtain the following Lemma.

Lemma 11 *For any $\beta \in (1/2, 1)$, that may depend on n , we have*

$$\frac{1}{1 - ((1 - \beta)/\beta)^N} \frac{2\beta - 1}{\beta} \geq \mathbf{P}_N(T_0 < T_N^+) \geq \frac{2\beta - 1}{4}. \quad (25)$$

In particular, if $\varepsilon \geq c/n$ then

$$\mathbf{P}_N(T_0 < T_N^+) = \Theta(\varepsilon). \quad (26)$$

Proof. The bounds in (25) are from Lemma 9. The upper bound of (26) comes from

$$\left(\frac{1 - \beta}{\beta}\right)^N = \left(\frac{1 - 2\varepsilon}{1 + 2\varepsilon}\right)^N \leq \exp\left(-\frac{4\varepsilon N}{1 + 2\varepsilon}\right) \leq \exp(-2\varepsilon N) \leq e^{-c}.$$

It follows that the first term on the left hand side of (25) is at most $1/(1 - e^{-c})$, which is a positive constant for any $c > 0$ constant or tending to infinity. \square

Lemma 12 *For $\varepsilon n = \Omega(1)$ and $\varepsilon \leq 1/2$ we have*

$$\pi(0) = \Theta\left(\sqrt{\frac{\varepsilon}{n}}\right).$$

Proof. Using that $p_i = 1/2 - 2\varepsilon i/n$ and $q_i = 1 - p_i$, for i constant $p_i, q_i \sim 1/2$. Thus combined with $p_0 = q_N = 1$, equation (7), tells us that

$$\pi(0)^{-1} = 1 + \frac{p_0}{q_1} + \sum_{j=2}^N \frac{p_0 \cdots p_{j-1}}{q_1 \cdots q_j} = \Theta\left(\sum_{j=1}^{N-1} \frac{p_1 \cdots p_j}{q_1 \cdots q_j}\right) \quad (27)$$

From $p_i = 1/2 - 2\varepsilon i/n$ and $q_i = 1/2 + 2\varepsilon i/n$, we get $p_i/q_i = 1 - \frac{8\varepsilon i/n}{1 + 4\varepsilon i/n}$. Thus, as $i \leq N = n/2$, and $0 < \varepsilon \leq 1/2$,

$$1 - \frac{8\varepsilon i}{n} \leq \frac{p_i}{q_i} = 1 - \frac{8\varepsilon i}{n + 4\varepsilon i} \leq 1 - \frac{4\varepsilon i}{n}.$$

For $|x| < 1$, $\exp -x/(1-x) \leq 1-x \leq \exp -x$, so

$$\exp\left(-\frac{8\varepsilon i}{n-8\varepsilon i}\right) \leq \frac{p_i}{q_i} \leq \exp\left(-\frac{4\varepsilon i}{n}\right).$$

Provided $i \leq N/4$, $p_i/q_i \geq \exp(-16\varepsilon i/n)$, and thus

$$\sum_{j=1}^{N/4} \prod_{i=1}^j \exp\left(-\frac{16\varepsilon i}{n}\right) \leq \pi(0)^{-1} \leq 2 \sum_{j=1}^{N/2} \prod_{i=1}^j \exp\left(-\frac{4\varepsilon i}{n}\right), \quad (28)$$

or equivalently

$$\pi(0)^{-1} = \Theta(1) \sum_{j=1}^{N/2} \exp\left(-\Theta(\varepsilon) \frac{j^2}{n}\right). \quad (29)$$

To finish the proof we need to check the above sum is $\sqrt{n/\Theta(\varepsilon)}$

$$\begin{aligned} \sum_{j=1}^{N/2} \exp\left(-\Theta(\varepsilon) \frac{j^2}{n}\right) &= \int_0^{N/2} \exp\left(-\Theta(\varepsilon) \frac{x^2}{n}\right) dx = \sqrt{\frac{n}{\Theta(\varepsilon)}} \int_0^{\Theta(\sqrt{\varepsilon n})} \exp(-x^2) dx \\ &= \sqrt{\frac{n}{\Theta(\varepsilon)}} \Phi(\Theta(\sqrt{\varepsilon n})). \end{aligned}$$

Observe that $\Phi(x)$ goes to $1/2$ as x goes to infinity and goes to 0 when $x \rightarrow 0$. Hence, if $\sqrt{\varepsilon n} = \Omega(1)$ then $\Phi(\Theta(\sqrt{\varepsilon n})) = \Theta(1)$.

□

6.2 Estimation of $\pi(N)$

We obtain $\pi(N)$ in terms of $\pi(0)$. By equation (6) we have that

$$\pi(N) = \pi(0) \frac{p_0 \cdots p_{N-1}}{q_1 \cdots q_{N-1}} = \pi(0) R,$$

say. Note that $p_0 = 1$ and using $N = n/2$, $\beta = 1 - \delta$

$$p_i = \frac{1}{2} + (1 - 2\beta) \frac{i}{n} = \frac{1}{2N} (N - (1 - 2\delta)i).$$

In this way we can write R as

$$R = \prod_{i=1}^{N-1} \frac{N - (1 - 2\delta)i}{N + (1 - 2\delta)i}. \quad (30)$$

Lemma 13 (i) Let $c > 1$ be a positive constant. If $1/(cn) \leq \delta < 1/2$, then

$$R = \Theta(1) \frac{1}{\sqrt{\delta}} \left(\frac{1}{2(1-\delta)^{1-\delta}\delta^\delta} \right)^{n/(1-2\delta)}. \quad (31)$$

(ii) Let c be a positive constant. If $\delta \leq 1/(cn)$, then

$$R = \Theta(1) \frac{\sqrt{n}}{2^n} \exp(\delta n \log n). \quad (32)$$

Proof. Case of (31). Let $\alpha = 1/(1 - 2\delta)$, then from (30) we can write R as

$$\begin{aligned} R &= \prod_{k=1}^{N-1} \frac{\alpha N - k}{\alpha N + k} = \frac{(\alpha N - 1) \cdots (\alpha N - (N - 1))}{(\alpha N + 1) \cdots (\alpha N + (N - 1))} \\ &= \frac{\alpha}{\alpha - 1} \frac{\Gamma(\alpha N) \Gamma(\alpha N)}{\Gamma((\alpha + 1)N) \Gamma((\alpha - 1)N)}, \end{aligned}$$

where $\Gamma(z) = (z - 1)\Gamma(z - 1)$. Provided z is at least a small positive constant we can an asymptotic expansion of the Gamma function as given by

$$\Gamma(z) = \sqrt{2\pi} z^{z-1/2} e^{-z} \left(1 + \mathcal{O}\left(\frac{1}{z}\right) \right).$$

Thus, simplifying extensively

$$\begin{aligned} R &= \Theta(1) \sqrt{\frac{\alpha + 1}{\alpha - 1}} \left(\frac{\alpha^{2\alpha}}{(\alpha + 1)^{\alpha+1} (\alpha - 1)^{\alpha-1}} \right)^N \\ &= \Theta(1) \sqrt{\frac{\alpha + 1}{\alpha - 1}} \left(\left(\frac{\alpha}{\alpha + 1} \right)^{\alpha+1} \left(\frac{\alpha}{\alpha - 1} \right)^{\alpha-1} \right)^N \\ &= \Theta(1) \sqrt{\frac{1 - \delta}{\delta}} \left(\left(\frac{1}{2 - 2\delta} \right)^{1-\delta} \left(\frac{1}{2\delta} \right)^{\delta} \right)^{2N/(1-2\delta)}. \end{aligned}$$

Equation (31) follows directly from this.

Case of (32).

We use the expression for R given in (30). This is equivalent to

$$R = \prod_{k=1}^{N-1} \frac{N - k}{N + k} \prod_{k=1}^{N-1} \frac{1 + 2\delta k/(N - k)}{1 - 2\delta k/(N + k)}. \quad (33)$$

The first product can be written as

$$\prod_{k=1}^{N-1} \frac{N - k}{N + k} = 2 \frac{N!N!}{(2N)!} = \Theta(1) \frac{\sqrt{n}}{2^n}.$$

If $\delta < 1/cn$ then the denominator in the second product of (33) is $1 - \mathcal{O}(\delta)$ For the numerator, $2\delta k/(N - k) < 1$, so

$$1 + 2\delta k/(N - k) = \exp \left(2\delta \frac{k}{N - k} - \mathcal{O} \left(\frac{\delta k}{N - k} \right)^2 \right).$$

Here

$$\sum_{k=1}^{N-1} \frac{k}{N-k} = \mathcal{O}(N) + N \log N,$$

and

$$\delta^2 \sum_{k=1}^{N-1} \left(\frac{k}{N-k} \right)^2 = \mathcal{O}(\delta^2 N^2) = \mathcal{O}(1).$$

Thus

$$\prod_{k=1}^{N-1} \frac{1 + 2\delta k/(N-k)}{1 - 2\delta k/(N+k)} = \Theta(1 + \delta) \exp(\delta n \log n),$$

completing the proof of (32). \square

6.3 Proof of Theorem 2

We apply the various results we have accumulated so far to

$$\mathbf{E}_0 T_N = \Theta(1) \frac{1}{\pi(N) \mathbf{P}_N(T_0 < T_N^+)} \quad (34)$$

and

$$\pi(N) = \pi(0)R. \quad (35)$$

Here $\mathbf{P}_N(T_0 < T_N^+)$ is given by Lemma 10, $\pi(0)$ by Lemma 12, and R by Lemma 13.

From (31) with $\beta = 1/2 + \varepsilon = 1 - \delta$ (i.e. $\delta = 1/2 - \varepsilon, 1 - 2\delta = 2\varepsilon$) we obtain

$$R = \Theta(1) \left((1 + 2\varepsilon)^{1+2\varepsilon} (1 - 2\varepsilon)^{1-2\varepsilon} \right)^{-n/4\varepsilon}. \quad (36)$$

For $x \leq 1/2$,

$$(1+x)^{1+x} (1-x)^{1-x} = \exp \left(\sum_{\substack{j \geq 2 \\ j \text{ even}}} \frac{2}{(j-1)j} x^j \right). \quad (37)$$

Put $x = 2\varepsilon$, and use (37) in (36) to give

$$R = \Theta(1) \exp \left(-\frac{n}{4\varepsilon} \sum_{\substack{j \geq 2 \\ j \text{ even}}} \frac{2}{(j-1)j} (2\varepsilon)^j \right). \quad (38)$$

From (25) we get

$$\Theta(\varepsilon) \frac{1}{1 - e^{-\Theta(n\varepsilon)}} \geq \mathbf{P}_N(T_0 < T_N^+) \geq \Theta(\varepsilon). \quad (39)$$

Theorem 2, Part i. Case $0 \leq \varepsilon \leq c/n$.

It follows from Lemma 4 and Theorem 2 iii that for any $\beta = 1/2 + \varepsilon$, that $\mathbf{E}_0 T_N = \Omega(n^2)$. We next prove that $\mathbf{E}_0 T_N = \mathcal{O}(n^2)$. Suppose that $\varepsilon = c/n$ where $c > 0$ is constant. From (38),

$$\pi(N) = \pi(0)R = \pi(0)\Theta(e^{-c}),$$

In consequence $\pi(N) = \Theta(\pi(0))$. For $\beta \geq 1/2$ it follows from $\pi(j+1) = \pi(j)p_j/q_{j+1}$ that $\pi(j) \geq \pi(j+1)$. Thus $\pi(j) = \Theta(1/n)$ for all j . Thus from (34) and the right hand side of (39)

$$\mathbf{E}_0 T_N = \frac{1}{\pi(N)\mathbf{P}_N(T_0 < T_N^+)} = \mathcal{O}\left(\frac{n}{\varepsilon}\right) = \mathcal{O}\left(\frac{n^2}{c}\right).$$

Thus for any c constant and $\varepsilon \leq c/n$, Lemma 4 we have that $\mathbf{E}_0 T_N = \mathcal{O}(n^2)$. Combining this with the result (from above) that $\mathbf{E}_0 T_N = \Omega(n^2)$ gives the result that $\mathbf{E}_0 T_N = \Theta(n^2)$.

Theorem 2, Part ii. Case $c/n \leq \varepsilon = o(1)$.

From Lemma 12 we have

$$\pi(0) = \Theta(\sqrt{\varepsilon/n}), \quad (40)$$

and (39) gives $\mathbf{P}_N(T_0 < T_N^+) = \Theta(\varepsilon)$. Combining this with (38) gives

$$\mathbf{E}_0 T_N = \Theta(1) \frac{\sqrt{n}}{\varepsilon^{3/2}} \exp\left(-\frac{n}{4\varepsilon} \sum_{\substack{j \geq 2 \\ j \text{ even}}} \frac{2}{(j-1)j} (2\varepsilon)^j\right) = \Theta(1) \frac{\sqrt{n}}{\varepsilon^{3/2}} \exp(n\varepsilon(1 + \mathcal{O}(\varepsilon^2))).$$

For the asymptotic we used that $\varepsilon \leq 1/2$ and $\sum_{J \geq 1} 1/(J(J+1)) = 1$.

Theorem 2: Parts iii and iv.

This is an application of (31), (39) and (40) to (34). Note that $\varepsilon > 0$ constant.

Theorem 2: Parts v and vi.

This is an application of (32), (39) and (40) to (34) giving

$$\mathbf{E}_0 T_N = \Theta(\sqrt{n}) \frac{2^n}{\sqrt{n}} \exp(-\delta n \log n).$$

When $\delta = \mathcal{O}(1/n \log n)$ the term $\exp(-\delta n \log n)$ is constant giving the proof of vi.

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